

# Asset Pricing Models

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June 7, 2018

# Broad Lesson Plan

- 1 Introduction
- 2 CAPM
- 3 Lemma
- 4 Arbitrage Pricing Theory
- 5 What Factors?
- 6 Takeaways

# Background

- Intuitively, a single market measure such as beta cannot capture all the information relevant to the price of a stock.
- Roll and Ross state that a security's long-run return is a function of changes in
  - \* Inflation
  - \* Industrial production
  - \* Risk premiums
  - \* The slope of the term structure of interest rates
- Generalization of Capital Asset Pricing Model (CAPM) is needed.

# Portfolio Construction, Expected Return

- Consider a portfolio with  $w$  portion invested in an asset  $i$  of expected return  $r_i := \mathbb{E}(r_{i,t})$  and  $1 - w$  portion invested in the market portfolio of expected return  $r_m := \mathbb{E}(r_{m,t})$ .
- The return of this portfolio, denoted by  $r_{w,t}$ , is a weighted combination of  $r_{i,t}$  and  $r_{m,t}$ .

$$r_{w,t} = wr_{i,t} + (1 - w)r_{m,t} \quad (1)$$

- By the linear property of the expectation operator  $\mathbb{E}(\cdot)$ , the expected return of this portfolio is

$$r_w = wr_i + (1 - w)r_m. \quad (2)$$

## Lemma: Variance of $aX + bY$

▢ Let  $\mu_X = \mathbb{E}(X)$ ,  $\mu_Y = \mathbb{E}(Y)$ , and  $\mu_Z = \mathbb{E}(Z)$ .

▢ Let  $Z := aX + bY$ .

▢  $\mu_Z = \mathbb{E}(Z) = a \mathbb{E}(X) + b \mathbb{E}(Y) = a\mu_X + b\mu_Y$

▢ The definition of variance of is

$$\begin{aligned}
 \mathbb{V}(Z) &= \mathbb{E}((Z - \mu_Z)^2) = \mathbb{E}((aX - a\mu_X + bY - b\mu_Y)^2) \\
 &= \mathbb{E}((aX - a\mu_X)^2 + (bY - b\mu_Y)^2 + 2(aX - a\mu_X)(bY - b\mu_Y)) \\
 &= a^2 \mathbb{E}((X - \mu_X)^2) + b^2 \mathbb{E}((Y - \mu_Y)^2) \\
 &\quad + 2ab \mathbb{E}((X - \mu_X)(Y - \mu_Y)) \\
 &= a^2 \mathbb{V}(X) + b^2 \mathbb{V}(Y) + 2ab \mathbb{C}(X, Y)
 \end{aligned}$$

## Variance of Portfolio's Return

▢ To (1), apply the variance operator  $\mathbb{V}(\cdot)$ . Using the lemma, we get

$$\mathbb{V}(r_{w,t}) = w^2 \mathbb{V}(r_{i,t}) + (1-w)^2 \mathbb{V}(r_{m,t}) + 2w(1-w) \mathbb{C}(r_{i,t}, r_{m,t}).$$

▢ For convenience, we denote

$$\S \sigma_w^2 := \mathbb{V}(r_{w,t}), \quad \sigma_i^2 := \mathbb{V}(r_{i,t}) \quad \text{and} \quad \sigma_m^2 = \mathbb{V}(r_{m,t})$$

§ The covariance  $\sigma_{im} := \mathbb{C}(r_{i,t}, r_{m,t})$ .

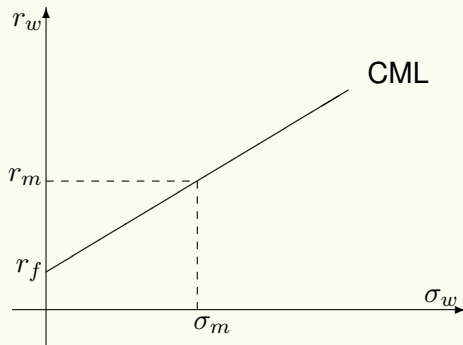
▢ With these notations, the variance  $\mathbb{V}(r_{w,t})$  simplifies to

$$\sigma_w^2 = w^2 \sigma_i^2 + 2w(1-w) \sigma_{im} + (1-w)^2 \sigma_m^2 \quad (3)$$

# Slope of CML

- 1 The slope of the CML is the Sharpe ratio. At  $w = 0$ , or for  $\sigma_m$ , we have

$$\frac{r_m - r_f}{\sigma_m} = \left. \frac{dr_w}{d\sigma_w} \right|_{w=0}$$



## Derivation of Slope

- 1 It is tedious to compute  $\frac{dr_w}{d\sigma_w}$  directly.
- 2 Instead, we have, by chain rule,

$$\frac{dr_w}{d\sigma_w} = \frac{\frac{dr_w}{dw}}{\frac{d\sigma_w}{dw}}$$

- 3 From (2), we obtain  $\frac{dr_w}{dw} = r_i - r_m$
- 4 From (3), we obtain

$$2\sigma_w \frac{d\sigma_w}{dw} = 2w\sigma_i^2 + 2(1-2w)\sigma_{im}, -2(1-w)\sigma_m^2,$$

equivalently,

$$\frac{d\sigma_w}{dw} = \frac{w\sigma_i^2 + (1-2w)\sigma_{im} - (1-w)\sigma_m^2}{\sigma_w}.$$

## Slope at $w = 0$

- 1 Putting everything together,

$$\frac{dr_w}{d\sigma_w} = \frac{\frac{dr_w}{dw}}{\frac{d\sigma_w}{dw}} = \frac{r_i - r_m}{\frac{w\sigma_i^2 + (1 - 2w)\sigma_{im} - (1 - w)\sigma_m^2}{\sigma_w}}$$

- 2 At  $w = 0$ ,  $\sigma_w = \sigma_m$ . Moreover, given that the slope is the Sharpe ratio, we have

$$\frac{r_m - r_f}{\sigma_m} = \frac{r_i - r_m}{\left(\frac{\sigma_{im} - \sigma_m^2}{\sigma_m}\right)}$$

$$r_m - r_f = \frac{r_i - r_m}{\left(\frac{\sigma_{im} - \sigma_m^2}{\sigma_m^2}\right)} = \frac{r_i - r_m}{\left(\frac{\sigma_{im}}{\sigma_m^2} - 1\right)}$$

## Slope at $w = 0$ (Cont'd)

- 3 For any asset  $i$  that is not a market portfolio,  $\frac{\sigma_{im}}{\sigma_m^2} - 1 \neq 0$ . So we multiply it to both sides to obtain

$$(r_m - r_f) \left( \frac{\sigma_{im}}{\sigma_m^2} - 1 \right) = r_i - r_m$$

$$\frac{\sigma_{im}}{\sigma_m^2} (r_m - r_f) - (r_m - r_f) = r_i - r_m$$

- 4 Knowing that  $\frac{\sigma_{im}}{\sigma_m^2} = \beta_i$ , we write,

$$\beta_i (r_m - r_f) = (r_m - r_f) + r_i - r_m = r_i - r_f$$

- 5 Hence CAPM ensues:

$$r_i - r_f = \beta_i (r_m - r_f)$$



# A Lemma

## Lemma

Let  $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_K$  and  $\mathbf{z}$  be  $N$ -dimensional column vectors. Suppose another  $N$ -dimensional column vector,  $\mathbf{w}$ , is orthogonal to all these  $K + 1$  vectors. In other words,

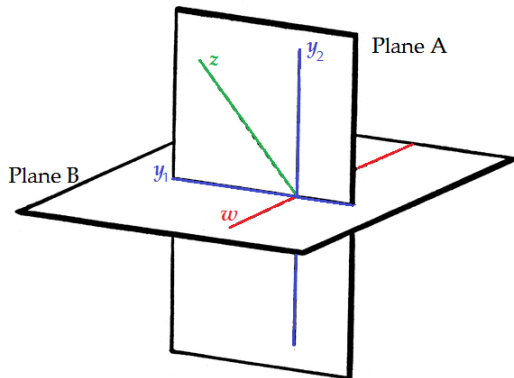
$$\mathbf{w}^\top \mathbf{z} = 0; \quad \mathbf{w}^\top \mathbf{y}_i = 0, \quad i = 1, 2, \dots, K.$$

Then,  $\mathbf{z}$  is a linear combination of  $\mathbf{y}_i, i = 1, 2, \dots, K$ :

$$\mathbf{z} = \sum_{i=1}^K \lambda_i \mathbf{y}_i,$$

where  $\lambda_i \neq 0$  for all  $i$ .

## Intuitive Illustration



- Plane A and Plane B are perpendicular.
- Any vector on Plane A is perpendicular to a vector  $w$  on Plane B.
- Clearly, vector  $z$  can be expressed as a linear combination of  $y_1$  and  $y_2$  on Plane A.

## Proof of Lemma

- Since  $\mathbf{w}^\top \mathbf{y}_i = 0$ ,  $i = 1, 2, \dots, K$ , for non-zero  $\lambda_i$ ,

$$\sum_{i=1}^K (-\lambda_i \mathbf{w}^\top \mathbf{y}_i) = 0.$$

- Adding  $\mathbf{w}^\top \mathbf{z} = 0$  to the above, we have  $\mathbf{w}^\top \mathbf{z} - \sum_{i=1}^K \lambda_i \mathbf{w}^\top \mathbf{y}_i = 0$ .

- Factoring out  $\mathbf{w}^\top$ , we obtain  $\mathbf{w}^\top \left( \mathbf{z} - \sum_{i=1}^K \lambda_i \mathbf{y}_i \right) = 0$ .

- Since  $\mathbf{w}$  is any arbitrary vector, it must be that

$$\mathbf{z} = \sum_{i=1}^K \lambda_i \mathbf{y}_i.$$

# APT Assumptions

- Assume that there are  $M$  risk factors  $F_k$ ,  $k = 1, 2, \dots, M$ .
- Without loss of generality, the means of these factors are all zero.
- Suppose there are  $N$  assets, each with return  $r_i$ , and

$$r_i = a_i + \sum_{k=1}^M b_{ik} F_k + e_i,$$

where  $a_i$  and  $b_{ik}$  for  $k = 1, 2, \dots, M$  are constants specific to the  $i^{\text{th}}$  asset.

- Motivated by the thinking of ordinary least squares, “explanatory variables”  $F_k$  should not have correlation with “noise”  $e_t$ .

# Portfolio Construction

- Suppose we can form a large portfolio of  $n_1$  assets with equal weight  $x_i = \frac{1}{n_1}$  such that  $\sum_{i=1}^{n_1} x_i = 1$ .

- By the **law of large numbers**,

$$\sum_{i=1}^{n_1} x_i e_i = \frac{1}{n_1} \sum_{i=1}^{n_1} e_i \approx 0.$$

- Then this portfolio's **expected return**  $R_1$  is

$$R_1 = A_1 + \sum_{k=1}^M B_{1k} F_k,$$

where  $A_1 = \sum_{i=1}^{n_1} x_i a_i$  and  $B_{1k} = \sum_{i=1}^{n_1} x_i b_{ik}$  for all  $k = 1, 2, \dots, M$ .

## Large Number of Portfolios

- Likewise, we form a second large portfolio of  $n_2$  assets (distinct from those in the first portfolio) with the same method. This portfolio's expected return is

$$R_2 = A_2 + \sum_{k=1}^M B_{2k} F_k,$$

where  $A_2 = \sum_{i=1}^{n_2} x_i a_i$ , and  $B_{2k} = \sum_{i=1}^{n_2} x_i b_{ik}$  for all  $k$

- Suppose we can form  $Q$  such almost equally weighted portfolios. Since each asset is used only once in portfolios construction,

$$\sum_{i=1}^Q n_i = N.$$

## A Portfolio of Portfolios

- For  $Q$  portfolios, we have  $\mathbf{R} := [R_1 \quad R_2 \quad \cdots \quad R_Q]^\top$ .
- Likewise, we collect the coefficients as vector  $\mathbf{A}$  and matrix  $\mathbf{B}$ :

$$\mathbf{A} := \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_Q \end{bmatrix} \quad \text{and} \quad \mathbf{B} := \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1M} \\ B_{21} & B_{22} & \cdots & B_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ B_{Q1} & B_{Q2} & \cdots & B_{QM} \end{bmatrix}.$$

- We want to construct a **self-financing** and **factor-neutral** portfolio from these  $Q$  portfolios.
- The weight vector of this portfolio of portfolios is denoted by  $\mathbf{w} := [w_1 \quad w_2 \quad \cdots \quad w_Q]^\top$ .

## Factor Neutrality

- Let  $\mathbf{1}$  be a vector with all elements being 1.
- Self-financing means **zero cash outlay**:

$$\mathbf{w}^\top \mathbf{1} = \sum_{i=1}^Q w_i = 0.$$

- Factor neutrality means **orthogonality**:

$$\mathbf{w}^\top \mathbf{B}_k = 0, \quad k = 1, 2, \dots, M,$$

where  $\mathbf{B}_k = [B_{1k} \quad B_{2k} \quad \dots \quad B_{Qk}]^\top$  is the  $k^{\text{th}}$  column of matrix  $\mathbf{B}$ .

## Expected Return on a Portfolio of Portfolios

- ↳ The portfolio of portfolios has the return given by

$$\begin{aligned}\mathbf{w}^\top \mathbf{R} &= \sum_{i=1}^Q w_i R_i = \sum_{i=1}^Q w_i \left( A_i + \sum_{j=1}^M B_{ij} F_j \right) \\ &= \sum_{i=1}^Q w_i A_i + \sum_{i=1}^Q \sum_{j=1}^M w_i B_{ij} F_j \\ &= \mathbf{w}^\top \mathbf{A} + \sum_{j=1}^M \sum_{i=1}^Q (w_i B_{ij}) F_j \\ &= \mathbf{w}^\top \mathbf{A} + \sum_{j=1}^M (\mathbf{w}^\top \mathbf{B}_k) F_j \\ &= \mathbf{w}^\top \mathbf{A}\end{aligned}$$

- ↳  $\mathbf{A}$  is a **vector of Jensen's alphas**.

## Input from Quantitative Finance

- But since there is no cash outlay, no-arbitrage principle implies that this portfolio return must be zero.

$$\mathbf{w}^\top \mathbf{R} = 0$$

- It follows that vector  $\mathbf{w}^\top$  is **orthogonal** to vectors  $\mathbf{R}$ ,  $\mathbf{1}$ ,  $\mathbf{B}_1$ ,  $\mathbf{B}_2$ , and so on to  $\mathbf{B}_M$ .
- By the lemma, for  $\lambda_j \neq 0, j = 0, 1, 2, \dots, M$ , we obtain a linear model of multiple ( $M$ ) factors  $F_j$ :

$$\mathbf{R} = \lambda_0 \mathbf{1} + \sum_{j=1}^M \lambda_j \mathbf{B}_j. \quad (4)$$

## Special Cases

- ↪ If we set all  $\lambda_j = 0$   $j \neq 0$ , then there is no risk and  $\lambda_0 = r_f$ , the risk-free rate.
- ↪  $M = 1$ , i.e. one-factor model. Suppose this factor is the market factor, i.e.,  $\lambda_1 = \mathbb{E}(R_m - r_f)$ , where  $R_m$  is the expected return on **market portfolio**. Then (4) becomes

$$R_{Q \times 1} = r_f 1_{Q \times 1} + \mathbb{E}(R_m - r_f) B_{Q \times 1},$$

which is CAPM.

- ↪ Thus, APT (4) is a general result. The values  $\lambda_j$ ,  $j \neq 0$  are the **price of risk**, or **risk premium** associated with the different factors  $F_j$ ,  $j = 1, 2, \dots, M$ .

# Factors in French's Data Library

 $R_m - R_f$ 

market risk premium  
(market factor)

SMB

size factor

HML

value factor

RMW

profitability factor

CMA

investment factor

UMD

momentum factor

STR

short-term reversal factor

LTR

long-term reversal factor

## Remarks

- Risk factors are expressed as returns in percent at the stated frequency (daily, weekly, **monthly**, annually)
  - ❑ Expected return is proportional to risk!
  - ❑ The proportional constant is the “beta”.
  - ❑ Beta is a measure of the sensitivity to risk factor.
- Factor calculations are based on **CRSP** and **CRSP/Compustat Merged Database**.
  - ❑ firms listed on NYSE, AMEX, and NASDAQ for U.S. factors
- Value factor may be redundant (Fama & French (2015)).

## Questions

- 🐟 What are the assumptions of arbitrage pricing theory?
- 🐟 Liquidity risk of trading is a genuine concern. Is it a non-redundant factor?
- 🐟 Is there an algorithm to construct a portfolio with no exposure whatsoever to risk?
- 🐟 Answer: One of the ways is to use the **algorithm of principal component analysis**.

# Takeaways

- 1 Arbitrage pricing theory is a one-period model that generalizes one-period CAPM.
- 2 It is based on the principle of no risk-free arbitrary opportunity.
  - If the portfolio (of portfolios) has no exposure to any risk factor, it will not “earn” its risk premium.
  - If the portfolio is constructed with zero cost and zero risk exposure, the expected return will have to be zero.
- 3 Duality of risk and expected return

Risk factor  $\longleftrightarrow$  risk Premium